

The van der Waals fluid and its role in cosmology

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Abstract

We analyze the properties of a generic cosmological fluid described by the van der Waals equation of state. Exact solutions for the energy density evolution are found as implicit functions of the scale factor for a flat Friedmann-Robertson-Walker space-time. The possible values of the free parameter in the van der Waals equation are selected *a posteriori*, in accordance with asymptotic behaviors that are physically relevant. The stability of the model against small perturbations is studied through the hydrodynamic perturbations of the fluid for the relevant cases. It is found that a van der Waals fluid seems appropriate to describe non-eternal inflationary scenarios.

1 Introduction

With the advent of the inflationary cosmology [1, 2] and the relatively recent discovery of the accelerated expansion of the Universe at late times [3, 4], the cosmological components enlarged and exotic equations of state were needed to model the acceleration of the cosmic expansion [5, 6, 7]. The Chaplygin gas type equation of state is frequently studied in the literature and used to describe standard matter and dark energy as a single fluid [8, 9, 10, 11, 12, 13, 14]. Such equations of state can be generated by a standard scalar field with a specific potential [8, 15] or a tachyon scalar field [16], which can be also applied in various other contexts [17, 18, 19].

In the concept of standard matter and dark energy as a single fluid, the van der Waals equation of state has received some attention and was investigated in the literature [20, 21, 22]. The van der Waals fluid as a cosmological constituent in a variety of other contexts was also investigated [23, 24, 25, 26, 27]. Due to the non-linearities of such equations of state, the cosmological dynamics is usually solved numerically and asymptotic analytic solutions can be obtained just in some special cases. The main feature explored in these models concerns the ability to reproduce the accelerated and matter-dominated ages with a single component. Further, depending on the characteristics of the model in question, the properties of the early and/or the late Universe may be possible to describe.

In this work, we do not impose any desired behavior on the van der Waals fluid. Instead we look for general analytical solutions of the energy density in a flat Friedmann-Robertson-Walker space-time, and so select the physically relevant cases. Once the suitable cases are selected, the corresponding stability is tested through the analysis of the hydrodynamic perturbations.

2 Basic equations

Let us consider a cosmological fluid that obeys the van der Waals equation of state, which is written in the following form [23]

$$p = \frac{8w\rho}{3 - \rho} - 3\rho^2, \quad (1)$$

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where p and ρ are the pressure and energy density of the fluid, respectively. The parameter w is a non-negative constant. This non-linear equation of state presents some peculiar qualitative features. It has a vertical asymptote in $\rho = 3$, where the pressure diverges, and its form approaches the form $p \propto \rho$ when ρ is sufficiently small while the pressure is persistently negative when ρ is large enough. Further, the pressure can eventually switch from a positive value to a negative one and vice versa. Some of these features are very compelling for cosmology and justify a more detailed investigation of the van der Waals fluid in this context.

In a flat Friedmann-Robertson-Walker (F-R-W) background, assuming that the perfect fluid conservation equation holds, we have

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \quad (2)$$

with the dot denoting time derivative and a the scale factor.

From equations (1) and (2), one obtains a differential equation for a and ρ , namely

$$a(3 - \rho)\dot{\rho} + 3\dot{a}\rho(3 + 8w - 10\rho + 3\rho^2) = 0, \quad (3)$$

which can be rewritten in the form

$$a(3 - \rho)\frac{d\rho}{da} + 3\rho(3 + 8w - 10\rho + 3\rho^2) = 0. \quad (4)$$

This differential equation furnishes the necessary information to determine the energy density evolution of the fluid.

By assuming that general relativity governs the large scale dynamics, the energy density of the van der Waals fluid has an effect on the cosmic evolution through the Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{\rho}{3}, \quad (5)$$

where H stands for the Hubble parameter.

3 Background solutions

With the above equations in hand, we can now try to obtain the functional behavior of the energy density of the van der Waals fluid in a flat F-R-W background. Equation (4) is separable in terms of functions of a and ρ , generating the following differential expression

$$\frac{da}{a} = \frac{1}{3} \frac{d\rho}{3 + 8w - 10\rho + 3\rho^2} - \frac{1}{\rho} \frac{d\rho}{3 + 8w - 10\rho + 3\rho^2}. \quad (6)$$

The corresponding integration furnishes

$$\ln a = \frac{1}{2(3 + 8w)} \ln \left| \frac{3 + 8w - 10\rho + 3\rho^2}{\rho^2} \right| + \frac{4(2w - 3)}{3(3 + 8w)} \int \frac{d\rho}{3 + 8w - 10\rho + 3\rho^2} + k, \quad (7)$$

where k is a constant of integration. The integral in the right hand side of the above expression contains a polynomial of the form $A\rho^2 + B\rho + C$, with $A = 3$, $B = -10$ and $C = 3 + 8w$, whose integration for $4AC - B^2 > 0$ gives

$$a = \frac{C_0}{\rho^{\frac{1}{3+8w}}} (3 + 8w - 10\rho + 3\rho^2)^{\frac{1}{2(3+8w)}} \exp \left(\frac{2(2w - 3) \arctan \left(\frac{3\rho - 5}{2\sqrt{6w - 4}} \right)}{3(3 + 8w)\sqrt{6w - 4}} \right). \quad (8)$$

Here C_0 is a constant and this solution is valid for $w > 2/3$.

In the case for $4AC - B^2 = 0$, the solution is

$$a = \frac{C_0}{\rho^{\frac{1}{3+8w}}} (3 + 8w - 10\rho + 3\rho^2)^{\frac{1}{2(3+8w)}} \exp \left(\frac{4(2w - 3)}{3(3 + 8w)(5 - 3\rho)} \right) \quad (9)$$

which is valid only for $w = 2/3$.

Finally, the solution corresponding to the case $4AC - B^2 < 0$ reads

$$a = \frac{C_0}{\rho^{\frac{1}{3+8w}}} (3 + 8w - 10\rho + 3\rho^2)^{\frac{1}{2(3+8w)}} \left(\frac{3\rho - 5 - 2\sqrt{4-6w}}{3\rho - 5 + 2\sqrt{4-6w}} \right)^{\frac{(2w-3)}{3(3+8w)\sqrt{4-6w}}}, \quad (10)$$

which holds for $0 < w < 2/3$.

All the above solutions have asymptotic roots, i.e. they allow values of ρ that satisfy $a \rightarrow 0$, meaning that the equation of state considered here is compatible with what is expected for the Universe at early times. As we can realize, expressions (8), (9) and (10) involve terms quadratic in ρ and one can expect two roots for each of them. However, there is only one root common to all solutions and another is a root only of (8) and (9). Let us see this in more details as follows.

Expressions (8), (9) and (10) satisfy

$$\rho \rightarrow \rho_0 = (5 - 2\sqrt{4-6w})/3, \quad \text{when } a \rightarrow 0. \quad (11)$$

So ρ_0 is an asymptotic root for all the above solutions. But notice that for $w > 2/3$, solution (8), we have that (11) is complex and must be discarded for physical purposes. Then the value given by (11) is physically valid for $0 < w \leq 2/3$ and is the asymptotic initial condition from which the Universe expands as ρ decreases. Here it is important to point out that such a property means that a van der Waals fluid does not have a divergent energy density when $a \rightarrow 0$, i.e. it does not present a big bang singularity. Interestingly enough, is the result from equation (1) when evaluated at $\rho = \rho_0 = (5 - 2\sqrt{4-6w})/3$, namely,

$$p = -\rho_0. \quad (12)$$

This result establishes that for a sufficiently small a the van der Waals dynamics for $0 < w \leq 2/3$, encompassed by (9) and (10), is close to the one of the cosmological constant. Hence, the equation of state $p \approx -\rho$ holds in the beginning of the expansion and we have a de Sitter-like inflation for $0 < w \leq 2/3$. One more important point to stress is that once we take initial conditions around (11), the energy density will never take values near the vertical asymptote, $\rho = 3$, given that $\rho_0 < 3$ and ρ always will decrease with the increase of a .

The second asymptotic root is

$$\rho \rightarrow \rho'_0 = (5 + 2\sqrt{4-6w})/3, \quad \text{when } a \rightarrow 0, \quad (13)$$

which holds only for (8) and (9) - solution (10) diverges for $\rho \rightarrow \rho'_0$. This root also produces a complex ρ for $w > 2/3$, but even for $w = 2/3$, solution (9), it is not physically motivated as an asymptotic initial condition because it leads to limiting equations of state that are not viable.

Now consider the dynamics in the range $0 < w \leq 2/3$, were the initial condition (11) holds and the energy density dilutes as the Universe expands. Then, from solutions (9) and (10) it is straightforward to see that when $\rho \rightarrow 0$, i.e. a is large, we have

$$\rho \propto a^{-(3+8w)} \implies a \propto t^{\frac{2}{3+8w}}. \quad (14)$$

This is in accordance with (1), from which results $p = 8w\rho/3$ in the referred limit.

As follows, we will analyze the cases of physical interest: $w = 0$ and $w = 1/8$.

1) Case $w = 0$

It corresponds to the equation of state $p = -3\rho^2$, see equation (1), which through (2) furnishes the differential expression

$$3 \frac{da}{a} = \frac{d\rho}{3\rho^2 - \rho}. \quad (15)$$

The respective integration renders

$$\rho = (c_0 a^3 + 3)^{-1}, \quad (16)$$

where c_0 is a constant of integration.

By taking the Friedmann equation, $H^2 = \rho/3$, and applying (16), we have the differential expression bellow

$$\sqrt{c_0 a^3 + 3} \frac{da}{a} = \frac{dt}{\sqrt{3}} \implies 2\sqrt{c_0 u^2 + 3} \frac{du}{u} = \sqrt{3} dt, \quad (17)$$

where the change of variable $u^2 = a^3$ was performed.

Integration of (17) gives the implicit time evolution of the scale factor, namely,

$$\sqrt{c_0 a^3 + 3} + \sqrt{3} \ln \left(\frac{a^{3/2}}{6 + 2\sqrt{3}(c_0 a^3 + 3)} \right) = \frac{\sqrt{3}}{2} t + C_1, \quad (18)$$

with C_1 being a constant of integration.

The above expression has the asymptotic behaviors

$$a \propto \exp \left(\frac{t}{3} \right), \quad (19)$$

for a sufficiently small a , and

$$a \propto t^{2/3}, \quad (20)$$

corresponding to a large a . Hence, this case describes an inflation followed by a matter-dominated era type dynamics.

2) Case $w = 1/8$

The range of validity of solution (10) comprises $w = 1/8$, from which we have

$$a = \frac{C_0}{\rho^{1/4}} (4 - 10\rho + 3\rho^2)^{1/8} \left(\frac{3\rho - 5 + \sqrt{13}}{3\rho - 5 - \sqrt{13}} \right)^{\frac{11}{24\sqrt{13}}}. \quad (21)$$

From the general result (11), this expression has the root $\rho \rightarrow \rho_0 = (5 - \sqrt{13})/3$, which is the initial energy density and corresponds to the equation of state $p = -\rho_0$, as stated by (12). Also from (21), when ρ is sufficiently diluted we have $\rho \propto a^{-4}$, i.e. a radiation-dominated era, as anticipated in (14). Thus this case describes an inflationary period followed by a radiation-dominated era.

4 Hydrodynamic perturbations

In this section we will develop the general equations to evaluate the small scalar perturbations in a flat F-R-W background for a van der Waals fluid. As the background fluid does not present anisotropy, the line element for small scalar perturbations in the longitudinal gauge is [28, 29, 30]

$$ds^2 = a^2 [(1 + 2\Phi) d\eta^2 - (1 - 2\Phi) \delta_{ij} dx^i dx^j], \quad (22)$$

where η is the conformal time, $d\eta = dt/a$, and Φ is the gauge-invariant perturbation potential.

By introducing the Mukhanov-Sasaki variables [29, 30], which are defined by

$$\sigma = \frac{\Phi}{\sqrt{\rho + p}}, \quad \chi = \left(\frac{1}{a} \right) \left(1 + \frac{p}{\rho} \right)^{-1/2}, \quad (23)$$

the adiabatic perturbations, in the general relativity context, are described by the following compact differential equation

$$\sigma'' - v_s^2 \nabla^2 \sigma - \left(\frac{\chi''}{\chi} \right) \sigma = 0, \quad (24)$$

where the prime represents derivative with respect to the conformal time and $v_s^2 = \partial p / \partial \rho$ is the speed of sound.

For a plane-wave solution of the form $\sigma(\eta) e^{-i\mathbf{k} \cdot \mathbf{x}}$, the *short-wavelength perturbations* hold the condition $v_s k \eta \gg 1$ and the last term of equation (24) can be neglected, furnishing

$$\sigma'' + k^2 v_s^2 \sigma = 0. \quad (25)$$

The corresponding solution, valid for a sufficient slowly varying speed of sound (WKB approximation) reads

$$\sigma(\eta) \simeq \frac{\sigma_0}{\sqrt{v_s}} \exp \left(\pm i k \int v_s d\eta \right), \quad (26)$$

where σ_0 is a constant of integration.

For *long-wavelength perturbations*, when the condition $v_s k \eta \ll 1$ is valid, the spatial derivatives of (24) can be neglected and the following solution holds

$$\sigma(\eta) \simeq \sigma_0 \chi \int \frac{d\eta}{\chi^2}. \quad (27)$$

All the quantities appearing in the asymptotic solutions above can be expressed in a compact form as functions of the energy density ρ . The Friedmann equation furnishes the infinitesimal conformal time interval in the form

$$d\eta = \frac{dt}{a} = \sqrt{3} \frac{da}{a^2 \sqrt{\rho}}. \quad (28)$$

On the other hand, by using equation of state (1) one directly determines the speed of sound, namely,

$$v_s = \sqrt{\frac{\partial p}{\partial \rho}} = \sqrt{\frac{24w}{(3-\rho)^2} - 6\rho}. \quad (29)$$

Also equation (1) allows to write the variable χ given by (23)₂ as a function of ρ in the form

$$\chi = \frac{1}{a} \left[\frac{3 + 8w - 10\rho + 3\rho^2}{3 - \rho} \right]^{-1/2}. \quad (30)$$

Furthermore, the variable χ is also equivalent to

$$\chi = \frac{1}{a} \left[\frac{2}{3} \left(1 - \frac{\mathcal{H}'}{\mathcal{H}^2} \right) \right]^{-1/2}, \quad (31)$$

where $\mathcal{H} = a'/a$ is the Hubble parameter in terms of the conformal time. To get (31) from (30) one uses the acceleration equation

$$\frac{\mathcal{H}'}{a^2} = -\frac{1}{6}(\rho + 3p), \quad (32)$$

along with the equation of state (1). From the variable χ in the form (31) one can write the integral in (27) as

$$\int \frac{d\eta}{\chi^2} = \frac{2}{3} \left(\frac{a^2}{\mathcal{H}^2} - \int a^2 d\eta \right), \quad (33)$$

through an integration by parts. Using this result in (27) and returning to (23)₁, with a plane-wave solution of the form $\Phi(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}$, finally we have

$$\begin{aligned} \Phi(\eta) &= \sigma_1 \left(1 - \frac{\mathcal{H}}{a^2} \int a^2 d\eta \right) = \sigma_1 \left(1 - \frac{1}{a} \sqrt{\frac{\rho}{3}} \int a^2 d\eta \right) \\ \Phi(\eta) &= \sigma_1 \left(1 - \frac{\sqrt{\rho}}{a} \int \frac{da}{\sqrt{\rho}} \right), \end{aligned} \quad (34)$$

where (28) was used and σ_1 is a constant defined by $\sigma_1 = 2\sigma_0/\sqrt{3}$. With the quantities (26), (28), (29), (30) and (34) in hand, we can evaluate the perturbations for any w and determine the density contrast. The perturbed dynamics for the cases of interest will be analyzed as follows.

Let us now apply the above results to the case 1, when $w = 0$, and case 2, when $w = 1/8$. We immediately infer from equation of state of the case 1, $p = -3\rho^2$, that the speed of sound (29) is always imaginary, which will generate perturbations with increasing amplitude all the time. Hence, the case 1 is unstable against small perturbations. The same occurs for the case 2 when a is small, since in this situation $p \approx -\rho$ and solution (26) will give a combination of real exponentials for short-wavelength perturbations. However, the equation of state for the case 2 rapidly evolves to $p = \rho/3$ (radiation) as ρ dilutes with inflation, so reaching a real speed of sound and it is expected that persistent growing instabilities are prevented. Thus from now on we are focused on the perturbations for the case with $w = 1/8$, which is in fact the most interesting one since it naturally describes inflation followed by a radiation-dominated era.

Substituting (21) into (34), we get the perturbation potential for $w = 1/8$ in the long-wavelength limit as a function only of ρ , namely,

$$\Phi(\rho) = 1 - \rho^{3/4} (4 - 10\rho + 3\rho^2)^{-1/8} \left(\frac{3\rho - 5 - \sqrt{13}}{3\rho - 5 + \sqrt{13}} \right)^{\frac{11}{24\sqrt{13}}} \int_{\rho_0}^{\rho} f(\rho) d\rho, \quad (35)$$

where

$$f(\rho) \doteq \frac{1}{\sqrt{\rho}} \frac{da}{d\rho} = \frac{(\rho - 3) \left(\frac{3\rho - 5 + \sqrt{13}}{3\rho - 5 - \sqrt{13}} \right)^{\frac{11}{24\sqrt{13}}}}{3\rho^{7/4} (4 - 10\rho + 3\rho^2)^{7/8}}, \quad (36)$$

and ρ_0 denotes the initial energy density corresponding to a_0 . Here the constants C_0 and σ_1 are set to unity without loss of generality.

With this result in hand, using the following perturbation equation from the perturbed Einstein's equations

$$\nabla^2 \Phi - 3\mathcal{H}(\Phi' + \mathcal{H}\Phi) = \frac{a^2}{2} \delta\rho, \quad (37)$$

where $\delta\rho$ is the gauge-invariant density perturbation, we can now infer the density contrast.

Since we are analyzing the long-wavelength limit, the first term of (37) can be neglected (spatial derivatives). Further, using the Friedmann equation in terms of the conformal time, $3\mathcal{H}^2 = a^2\rho$, equation (37) furnishes

$$\frac{\delta\rho}{\rho} = -2 \left(\frac{\Phi'}{\mathcal{H}} + \Phi \right), \quad (38)$$

which gives the density contrast if Φ is known.

If we derive (34) with respect to conformal time, one has

$$\Phi'(\eta) = \frac{1}{a^2} (2\mathcal{H}^2 - \mathcal{H}') \int a^2 d\eta - \mathcal{H}. \quad (39)$$

Now we use (31) to eliminate \mathcal{H}' in the last expression, which is then substituted into (38). The corresponding result can be expressed in terms of ρ by means of (28) and (30). So, by considering the density perturbation of the form $\delta\rho(\eta)e^{-i\mathbf{k}\cdot\mathbf{x}}$, one obtains

$$\frac{\delta\rho}{\rho}(\rho) = -3\rho^{3/4} \frac{(4 - 10\rho + 3\rho^2)^{7/8}}{(3 - \rho)} \left(\frac{3\rho - 5 - \sqrt{13}}{3\rho - 5 + \sqrt{13}} \right)^{\frac{11}{24\sqrt{13}}} \int_{\rho_0}^{\rho} f(\rho) d\rho. \quad (40)$$

Thus we have an expression that gives the density contrast for $w = 1/8$ as a function only of ρ .

The scale factor (21) and amplitudes of the density contrast (40) and perturbation potential (35) are plotted in the left frame of Fig. 1. By observing the corresponding curves, we can see that the amplitude of the perturbation potential increases slowly with the dilution of ρ , i.e. with increasing a . On the other hand, the amplitude of the density contrast increases initially with the dilution of ρ and starts to decrease when ρ is sufficiently diluted, i.e. when a is sufficiently large and a radiation-dominated era is expected.

An alternative visualization of the perturbations is shown in the right frame Fig. 1, where the amplitudes of the perturbation potential (35) and density contrast (40) are expressed as functions of the scale factor. From this figure it is clear to see the slow growth of the amplitude of the perturbation potential with the cosmic expansion while the amplitude of the density contrast grows in the beginning and starts to decrease after a certain point of the cosmic expansion.

In view of these results, the van der Waals fluid could describe inflation followed by a radiation era, i.e. a non-eternal inflation which is stable against small perturbations in the limit of the long-wave perturbations, as shown by the above analysis of the hydrodynamic perturbations.

5 Concluding summary

In this work, we studied a generic cosmological fluid described by the van der Waals equation of state for a flat F-R-W space-time. General background solutions for the energy density of the fluid were obtained in analytical

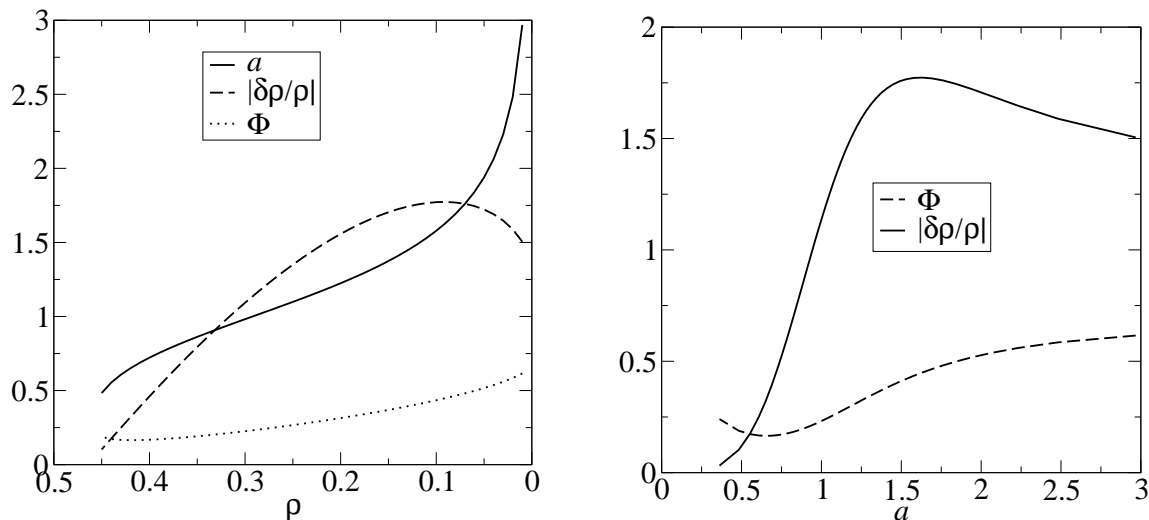


Figure 1: Left: The amplitudes of the perturbation potential and density contrast as functions of the scale factor ($w = 1/8$). Right: The scale factor and amplitudes of the density contrast and perturbation potential as functions of the energy density ($w = 1/8$).

form. From these solutions it was possible to identify the physically relevant cases. It was found that in general the energy density of a cosmological van der Waals fluid does not diverge as the scale factor goes to zero (no big bang singularity) and behaves like a cosmological constant type component. Two cases of interest in cosmology were found and they correspond to the values 0 and $1/8$ for the parameter w of the equation of state. For $w = 0$ the background solution describes an inflation followed by a matter-dominated era while for $w = 1/8$ it describes an inflation followed by a radiation-dominated era.

The viability of the relevant background solutions is tested through the analysis of the hydrodynamics perturbations. The case for $w = 0$ is unstable against small perturbations since the speed of sound is persistently imaginary. On the other hand, the case for $w = 1/8$ is stable against small perturbations in the limit of long-wave perturbations. This entails that the van der Waals model presents some skill in describing a non-eternal inflation, i.e. an initial *quasi*-cosmological constant dynamics followed by a radiation-dominated era.

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